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GENERALIZED METHOD OF BOUNDARY LAYER FUNCTION FOR BISINGULARLY PERTURBED DIFFERENTIAL COLE EQUATION

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Abstract

Asymptotic expansion of the solution of the boundary value problem of a singularly perturbed differential Cole equation of the second order with a weakly turning point is constructed by using the method of generalized boundary layer function.

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1. Statement of the Problem

The following problem is considered:

$$\varepsilon y''(x) + \sqrt{x}\,y'(x) - y(x) = 0, \quad 0 < x < 1, \tag{1}$$

$$y(0) = a, \quad y(1) = b, \tag{2}$$

where $0 < \varepsilon << 1$ is a small parameter, $x \in [0, 1]$; a, b are given constants.

The problem (1)-(2) is bisingularly perturbed. The unperturbed equation

$$\sqrt{x}\,y'(x) - y(x) = 0, \quad 0 < x < 1,$$

is of order one, and the solution of this equation

$$y_0(x) = ce^{2\sqrt{x}}, \quad c - \text{const.}$$

is a non-smooth function in $[0, 1]$.

We note that the problem (1)-(2) was considered earlier in [1-3] by the Kaplun method in Kaplun [4], and the asymptotic expansion of the solution till order one by the small parameter was obtained. Also, an asymptotic of the solution was constructed by the method of structural matching [5]. In [6], the approximation of the solution of this problem is determined for small parameters.

In this paper, we obtain the asymptotic expansion of the solution of the problem (1)-(2) by the method of generalized boundary layer function [6-9] of all orders in small parameter.

2. Method of Generalized Boundary Layer Function

We seek asymptotic representation of the solution of the problem (1)-(2) in the form:

$$y(x) = \sum_{k=0}^{n} \varepsilon^k y_k(x) + \sum_{k=0}^{3(n+1)} \mu^k \pi_k(t) + R(x, \varepsilon). \tag{3}$$

Here $t = x/\mu^2$, $\varepsilon = \mu^3$, $R(x, \varepsilon)$ is the reminder term.

Inserting (3) into the equation (1), we have

$$\sum_{k=0}^{n}\varepsilon^k(\varepsilon y_k''(x)+\sqrt{x}y_k'(x)-y_k(x))+\frac{1}{\mu}(\pi_0''(t)+\sqrt{t}\pi_0'(t))$$

$$+\sum_{k=1}^{3(n+1)}\mu^{k-1}(\pi_k''(t)+\sqrt{t}\pi_k'(t)-\pi_{k-1}(t))-\mu^{3(n+1)}\pi_{3(n+1)}(t)$$

$$+\varepsilon R''(x,\varepsilon)+\sqrt{x}R'(x,\varepsilon)-R(x,\varepsilon)-\sum_{k=1}^{n+1}\varepsilon^k h_k(x)+\sum_{k=1}^{n+1}\mu^{3k}h_k(t\mu^2)=0. \quad (4)$$

By the method of generalized boundary layer function, we put the term $\sum_{k=0}^{n-1}\varepsilon^k h_k(x)$ into the equation. We choose functions $h_k(x)$.

Taking into account the boundary condition (2), from (4), we obtain:

$$\sqrt{x}y_0'(x)-y_0(x)=0,\quad 0<x<1,\quad y_0(1)=b, \quad (5)$$

$$\sqrt{x}y_k'(x)-y_k(x)=h_{k-1}(x)-y_{k-1}''(x),\quad 0<x<1,\quad k\in N,\quad y_k(1)=0. \quad (6)$$

The solution of the problem (5)-(6) exists. It is unique and has the form

$$y_0(x)=be^{2(\sqrt{x}-1)},\quad y_k(x)=e^{2\sqrt{x}}\int_1^x\frac{h_{k-1}(s)-y_{k-1}''(s)}{\sqrt{s}}e^{-2\sqrt{s}}ds,\quad k\in N.$$

We choose indefinite functions $h_k(x)$ as follows:

$$y_{k-1}''(x)-h_{k-1}(x)\in C[0,1].$$

We can represent

$$y_0(x)=be^{-2}\left(1+2\sqrt{x}+\frac{(2\sqrt{x})^2}{2!}+\frac{(2\sqrt{x})^3}{3!}+\frac{(2\sqrt{x})^4}{4!}+\cdots+\frac{(2\sqrt{x})^n}{n!}+\cdots\right).$$

Let

$$h_1(x)=be^{-2}\left(2\sqrt{x}+\frac{(2\sqrt{x})^3}{3!}\right)''=-be^{-2}\left(\frac{1}{2\sqrt{x^3}}-\frac{1}{\sqrt{x}}\right).$$

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Then

$$y_0''(x) - h_0(x) \in C[0, 1], \quad \mu^3 h_1(t\mu^2) = -c_1\left(\frac{1}{2\sqrt{t^3}} - \frac{\mu^2}{\sqrt{t}}\right), \quad c_1 = be^{-2},$$

$$y_1(x) = c_1 e^{2\sqrt{x}} \int_1^x \left(-\frac{1}{2s^2} + \frac{1}{s} + \frac{1}{2s^2} e^{2\sqrt{s}} - \frac{1}{\sqrt{s^3}} e^{2\sqrt{s}}\right) e^{-2\sqrt{s}} ds.$$

We can rewrite $y_1(x)$ in the form:

$$y_1(x) = y_{1,0} + y_{1,1}(2\sqrt{x}) + y_{1,2}(2\sqrt{x})^2 + y_{1,3}(2\sqrt{x})^3 + \cdots,$$

where

$$y_{1,0} = \left(\frac{3}{2} + \frac{1}{2e^2}\right)c_1, \quad y_{1,1} = \left(\frac{1}{6} + \frac{1}{2e^2}\right)c_1,$$

$$y_{1,2} = \left(\frac{-1}{6} + \frac{1}{4e^2}\right)c_1, \quad y_{1,3} = \left(\frac{-1}{10} + \frac{1}{12e^2}\right)c_1.$$

Analogously, we have obtained

$$h_1(x) = (y_{1,1}(2\sqrt{x}) + y_{1,3}(2\sqrt{x})^3)'' = -\frac{y_{1,1}}{2\sqrt{x^3}} + \frac{6y_{1,3}}{\sqrt{x}}.$$

Then

$$y_2''(x) - h_2(x) \in C[0, 1], \quad \mu^6 h_2(t\mu^2) = -\frac{\mu^3 y_{1,1}}{2\sqrt{t^3}} + \frac{\mu^5 y_{1,3}}{\sqrt{t}}.$$

Continuing this process, we have

$$h_{k-1}(x) = -\frac{y_{k-1,1}}{2\sqrt{x^3}} + \frac{6y_{k-1,3}}{\sqrt{x}}, \quad k = 4, \ldots, n,$$

where $y_{k-1,1}$, $y_{k-1,3}$ are corresponding coefficients of the expansion of $y_{k-1}(x)$ in powers of $(2\sqrt{x})$.

From (4), we have following equations for the boundary functions $\pi_k(t)$:

$$L\pi_0 \equiv \pi_0''(t) + \sqrt{t}\pi_0'(t) = 0, \quad 0 < t < \widetilde{\mu},$$

$$\pi_0(0) = a - y_0(0), \quad \pi_0(\widetilde{\mu}) = 0, \quad \widetilde{\mu} = 1/\mu^2, \tag{7}$$

$$L\pi_{3k+1}(t) = \pi_{3k}(t) + \frac{y_{k,1}}{2\sqrt{t^3}}, \quad 0 < t < \widetilde{\mu},$$

$$\pi_{3k+1}(0) = 0, \quad \pi_{3k+1}(\widetilde{\mu}) = 0, \quad k = 0, 1, \dots, n, \tag{8}$$

$$L\pi_{3k+2}(t) = \pi_{3k+1}(t), \quad 0 < t < \widetilde{\mu}, \quad \pi_{3k+2}(0) = 0,$$

$$\pi_{3k+2}(\widetilde{\mu}) = 0, \quad k = 0, 1, \dots, n, \tag{9}$$

$$L\pi_{3k+3}(t) = \pi_{3k+2}(t) - \frac{y_{k,3}}{\sqrt{t}}, \quad 0 < t < \widetilde{\mu},$$

$$\pi_{3k}(0) = -y_k(0), \quad \pi_{3k}(\widetilde{\mu}) = 0, \quad k = 0, 1, \dots, n-1, \tag{10}$$

$$L\pi_{3(n+1)}(t) = \pi_{3n+2}(t) - \frac{y_{n,3}}{\sqrt{t}}, \quad 0 < t < \widetilde{\mu},$$

$$\pi_{3n}(0) = 0, \quad \pi_{3n}(\widetilde{\mu}) = 0. \tag{11}$$

The solution of the problem (7) is represented in the form

$$\pi_0(t) = (a - be^{-2})A\int_t^{\widetilde{\mu}} e^{-\frac{2}{3}s^{3/2}} ds, \qquad A = \left(\int_0^{\widetilde{\mu}} e^{-\frac{2}{3}s^{3/2}} ds\right)^{-1}.$$

We note that $\pi_0(t)$ will exponentially decrease as $t \to \widetilde{\mu}$.

Lemma 1. *The equation* $Lz := z''(t) + \sqrt{t}\pi'(t) = 0$ *has two linearly independent solutions*:

$$Y(t) = 1 - X(t), \qquad X(t) = A\int_t^{\widetilde{\mu}} e^{-\frac{2}{3}s^{3/2}} ds \left(A\int_0^{\widetilde{\mu}} e^{-\frac{2}{3}s^{3/2}} ds = 1 \right)$$

and the general solution of this one is represented in the form $z(t) = c_1 Y(t) + c_2 X(t)$, *here* c_1, c_2 *are constants, and* $Y(t) = O(t)$, $t \to 0$, $0 < X(t) \le 1$,

$$X(t) = t^{-\frac{1}{2}} e^{-\frac{2}{3}t^{3/2}} \left(1 - \frac{1}{2} t^{-\frac{3}{2}} + \cdots + \frac{(-1)^n}{2^n} \prod_{k=1}^{n} 1 \cdot 4 \cdot \ldots \cdot (3k-2) t^{-\frac{3n}{2}} + \cdots \right),$$

$$t \to \tilde{\mu}. \tag{12}$$

Lemma 2. *The boundary problem* $Lz(t) = 0$, $z(0) = z(\tilde{\mu}) = 0$ *has only trivial solution.*

The proofs of both the lemmas are straightforward.

Theorem 1. *The problem*

$$Lz(t) = f(t), \quad z(0) = 0, \quad z(\tilde{\mu}) = 0 \tag{13}$$

has the unique solution in the form

$$z(t) = \int_0^{\tilde{\mu}} G(t, s) e^{\frac{2}{3}s^{3/2}} \cdot f(s) ds. \tag{14}$$

Here

$$G(t, s) = \begin{cases} -Y(t)X(s), & 0 \le t \le s, \\ -Y(s)X(t), & s \le t \le \tilde{\mu} \end{cases}$$

is the Green function, $f(t) \in C(0, \tilde{\mu}]$ *and* $f(t) = O(t^{-3/2})$, $t \to 0$.

Proof. We can check the fact that (14) satisfies the equation $Lz(t) = f(t)$, $0 < t < \tilde{\mu}$ directly by substitution.

By using $G(t, s)$, we rewrite $z(t)$ in the form:

$$z(t) = J_1(t) + J_2(t),$$

where

$$J_1(t) = -X(t) \int_0^t Y(s) e^{\frac{2}{3}s^{3/2}} f(s) ds, \quad J_2(t) = Y(t) \int_0^t X(s) e^{\frac{2}{3}s^{3/2}} f(s) ds.$$

We show that both J_1 and J_2 satisfy the boundary conditions.

(1) (a) If $t \to 0$, then $|Y(t)| \le lt$, $|f(t)| \le lt^{-3/2}$. Therefore,

$$|J_1(t)| \le \int_0^t ls^{-1/2} e^{\frac{2}{3}s^{3/2}} ds \le l\sqrt{t}, \quad t \to 0.$$

(b) $t \to \tilde{\mu}$,

$$|J_1(t)| \le lt^{-1/2} e^{-\frac{2}{3}t^{3/2}} \left[\int_0^1 |Y(s)| e^{\frac{2}{3}s^{3/2}} |f(s)| ds + \int_1^t |Y(s)| e^{\frac{2}{3}s^{3/2}} |f(s)| ds \right]$$

$$\le lt^{-1/2} e^{-\frac{2}{3}t^{3/2}} + lt^{-1/2} e^{-\frac{2}{3}t^{3/2}} \int_1^t e^{\frac{2}{3}s^{3/2}} s^{-3/2} ds$$

$$\le lt^{-1/2} e^{-\frac{2}{3}t^{3/2}} + lt^{-1} = O(t^{-1}), \quad t \to \tilde{\mu}.$$

(2) (a) $t \to 0$,

$$|J_2(t)| \le lt \left(\int_t^1 s^{-3/2} ds + \int_1^{\tilde{\mu}} s^{-2} e^{\frac{2}{3}s^{3/2} - \frac{2}{3}s^{3/2}} ds \right) = O(\sqrt{t}), \quad t \to 0.$$

(b) $t \to \tilde{\mu} : |J_2(t)| \le l \int_t^{\tilde{\mu}} s^{-2} e^{\frac{2}{3}s^{3/2} - \frac{2}{3}s^{3/2}} ds = O(t^{-1}),\ t \to \tilde{\mu}.$

Therefore, $z(t)$ satisfies the boundary conditions. The existence and uniqueness of the solution of the problem (8)-(11) follows from Theorem 1: $|\pi_k(t)| < l = \text{const}$, $t \in [0, \tilde{\mu}]$.

Lemma 3. *Asymptotical expansions of functions* $\pi_k(t)$, $t \to \tilde{\mu}$ $(k = 1, 2, 3, \ldots)$ *have the following forms*:

$$\pi_1(t) = -\frac{y_{0,1}}{2t} \left(1 + \frac{4}{5\sqrt{t^3}} + \frac{7}{4t^3} + \frac{42}{11\sqrt{t^9}} + \frac{39}{2t^7} + \cdots \right),$$

$$\pi_2(t) = \frac{y_{0,1}}{\sqrt{t}}\left(1 + \frac{23}{40\sqrt{t^3}} + \frac{173}{2t^3} + \cdots\right), \quad \pi_3(t) = -\frac{23y_{0,1}}{60\sqrt{t^3}} + O\left(\frac{1}{t^3}\right),$$

$$\pi_{3k+1}(t) = t^{-1}\sum_{j=0}^{\infty} l_{3k+1,\,j}\, t^{-\frac{3}{2}j}, \quad \pi_{3k+2}(t) = t^{-1/2}\sum_{j=0}^{\infty} l_{3k+2,\,j}\, t^{-\frac{3}{2}j},$$

$$\pi_{3k}(t) = \sum_{j=1}^{\infty} l_{3k,\,j}\, t^{-\frac{3}{2}j}.$$

Proofs of Lemma 3. First proof. We can prove this lemma by application of formula (12) and Theorem 1.

Second proof. We can get these representations from equations (8)-(11) directly.

Now, we will prove the boundedness of the remainder function $R(x, \varepsilon)$. This function satisfies the following equation:

$$\varepsilon R''(x, \varepsilon) + \sqrt{x}R'(x, \varepsilon) - R(x, \varepsilon) = \mu^{3(n+1)}\pi_{3(n+1)}(t) + \varepsilon^{n+1}(h_n(x) - y_n''(x)),$$

$$R(0, \varepsilon) = 0, \quad R(1, \varepsilon) = 0.$$

Applying Theorem 26.4 [10, p. 117] to this problem, we have

$$|R(x, \varepsilon)| \le \varepsilon^{n+1}C \max_{\substack{0\le x\le 1\\ 0\le t\le \tilde{\mu}}} |\pi_{3(n+1)}(t) + h_n(x) - y_n''(x)|.$$

Therefore, $R(x, \varepsilon) = O(\varepsilon^{n+1})$, $\varepsilon \to 0$, $x \in [0, 1]$.

Thus, we have

Theorem 2. *The asymptotical expansion of the solution of the problem* (1)-(2) *has the following form*:

$$y(x) = \sum_{k=0}^{n} \varepsilon^k y_k(x) + \sum_{k=0}^{3(n+1)} \mu^k \pi_k(t) + O(\varepsilon^{n+1}), \quad \varepsilon \to 0.$$

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Conclusion

An asymptotical expansion of the solution of the boundary value problem of bisingularly perturbed differential Cole equation is obtained by using the generalized method of the boundary layer function. An estimate of the remainder is also determined.

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