ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2023, Vol. 44, No. 12, pp. 5149–5157. © Pleiades Publishing, Ltd., 2023.

Boundary Value Problems for a Mixed Equation of Parabolic-Hyperbolic Type of the Third Order

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(Submitted by T. K. Yuldashev)

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Received August 13, 2023; revised September 14, 2023; accepted September 30, 2023

Abstract—In this article, the existence and uniqueness of solution of the conjugation problem in a rectangular domain for a third-order partial differential equation is proved, when the characteristic equation has 3 multiple roots for $y > 0$, and it has 1 simple and 2 multiple roots for $y < 0$. Using the Green's functions and the method of integral equations, the solution of the problem is equivalently reduced to solving the boundary value problem for the trace of the desired function at $y = 0$, and then to solving the Fredholm integral equation of the 2nd kind. The one-valued solvability of Fredholm integral equation of the 2nd kind is proved by the method of successive approximations. The solution of the problem for $y > 0$ is constructed by the Green's function method, and for $y < 0$ by reducing to the problem of a two-dimensional Volterra integral equation of the 2nd kind.

DOI: 10.1134/S1995080223120065

Keywords and phrases: *Differential equation, third order, multiple characteristics, conjugation problem, boundary conditions, uniqueness, existence, Green's function, integral equations.*

1. PROBLEM STATEMENTS

The theory of partial differential equations is one of the main directions in mathematical physics and mechanics. The study of many problems in gas dynamics, elasticity theory, and the theory of plates and shells leads to the consideration of high-order partial differential equations [1]. Third-order partial differential equations are considered when solving a number of problems in the theory of nonlinear acoustics, in the hydrodynamic theory of space plasma and fluid filtration in porous media, as well as wave propagation in weakly dispersive media, in cold plasma and hydrodynamics [2]. It is of interest to study boundary value problems when combining different types of equations.

Statement of the Problem 1. In the domain $D = \{(x,y) : 0 < x < \ell,\, -h_1 < y < h\}$ $(\ell,\, h,\, h_1 > 0)$ we consider an equation

$$L_1(u) \equiv u_{xxx} - u_{xy} = 0, \quad (x,y) \in D_1 = D \cap (y > 0), \tag{1}$$

$$L_2(u) \equiv u_{xxy} + au_x + bu_y + cu = 0, \quad (x,y) \in D_2 = D \cap (y < 0), \tag{2}$$

where $a,\ b,$ and c are given functions and

$$a(x,y),\, a_x(x,y),\, b(x,y),\, b_y(x,y),\, c(x,y) \in C(\bar{D}_2). \tag{3}$$

Problem 1. Find in the domain D the function $u(x,y)$ from the class $C(\bar{D}) \cap C^1(D) \cap \left[C^{3,1}(D_1) \cup C^{2,1}(D_2)\right]$, satisfying the equation (1) in the domain D_1 and the boundary conditions

$$u(0,y) = \varphi_1(y), \quad u(\ell,y) = \varphi_2(y), \quad 0 \le y \le h, \tag{4}$$

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$$u_x(0,y)=\varphi_3(y),\quad 0\le y\le h, \tag{5}$$

satisfying the equation (2) in the domain D_2 and the boundary conditions

$$u(0,y)=\psi_1(y),\quad u_x(0,y)=\psi_2(y),\quad -h_1\le y\le 0, \tag{6}$$

where $\varphi_i(y)$ $\left(i=\overline{1,\,3}\right)$, $\psi_j(y)$ $(j=1,2)$ are given functions and

$$\varphi_1(0)=\psi_1(0),\quad \varphi_3(0)=\psi_2(0). \tag{7}$$

The equation (1) in the region D_1 with respect to the highest derivative has a 3-fold characteristic $y=const$, and the equation (2) in the domain D_2 with respect to the highest derivative has one 2-multiple characteristic $y=const$ and one simple characteristic $x=const$. According to the classification from [3], the equation (1) corresponds to the first canonical form, and the equation (2) corresponds to the second canonical form. Consequently, in the domain D, when passing through the line $y=0$, the type of equation changes. Therefore, the equations (1) and (2) together are an equation of mixed type [4]. The equation (1) in [5] is called an equation with multiple characteristics and in this work [5] the various boundary value problems for mixed equation are studied. We note, that various boundary value problems for third- and high-order equations are considered by many authors, for example, in [6–23].

2. REDUCING THE PROBLEM TO AN INTEGRAL EQUATION

From the formulation of Problem 1 it follows that the following gluing conditions are satisfied on the line $y=0$ [4]

$$\forall\, x\in[0,\ell]:\quad u(x,-0)=u(x,+0)=\tau(x),\quad u_y(x,-0)=u_y(x,+0)=\nu(x),$$

where $\tau(x)$ and $\nu(x)$ are yet unknown functions to be determined.

Passing to the limit at $y\to+0$ from the equation (1), we have the relation between the functions $\tau(x)$ and $\nu(x)$, obtained from the domain D_1

$$\tau'''(x)-\nu'(x)=0. \tag{8}$$

In addition, from the formulation of Problem 1 we have the following matching conditions

$$\tau(0)=\varphi_1(0),\quad \tau(\ell)=\varphi_2(0),\quad \tau'(0)=\varphi_3(0).$$

Passing to the limit at $y\to-0$ from the equation (2), we have the relation between the functions $\tau(x)$ and $\nu(x)$, obtained from the domain D_2

$$\nu''(x)+a(x,0)\tau'(x)+b(x,0)\nu(x)+c(x,0)\tau(x)=0. \tag{9}$$

Let's represent the equation (9) in the form

$$\nu''(x)+b(x,0)\nu(x)=\mathrm{T}(x), \tag{10}$$

where $\mathrm{T}(x)=-a(x,0)\tau'(x)-c(x,0)\tau(x)$. From the matching condition obtained from the region D_2, we have

$$\nu(0)=\psi_1'(0),\quad \nu'(0)=\psi_2'(0). \tag{11}$$

Using the integration method, we reduce the solution of the equation (10), satisfying the conditions (11), to the Volterra integral equation of the 2nd kind

$$\nu(x)=\int\limits_0^x K_1(x,\xi)\,\nu(\xi)d\xi+\int\limits_0^x(x-\xi)\,\mathrm{T}(\xi)d\xi+\psi(x), \tag{12}$$

where $K_1(x,\xi)=-(x-\xi)b(\xi,0)$, $\psi(x)=\psi_1'(0)+\psi_2'(0)x$.

We note that $\forall\,(x,\xi)\in[0,\ell]\times[0,\ell]$: $K_1(x,\xi)\in C([0,\ell]\times[0,\ell])$. By the symbol $\Gamma_1(x,\xi)$ we denote the resolvent of the kernel $K_1(x,\xi)$. Then, the solution to the equation (12) can be represented as

$$\nu(x)=\int\limits_0^x\Gamma(x,\xi)\,\mathrm{T}(\xi)d\xi+\Psi(x), \tag{13}$$

where

$$\Gamma(x,\xi) = x - \xi + \int\limits_{\xi}^{x} (t-\xi)\Gamma_1(x,t)dt,\ \ \Psi(x) = \psi(x) + \int\limits_{0}^{x} \Gamma_1(x,\xi)\psi(\xi)d\xi.$$

Next, substituting the value $\mathrm{T}(x)$ into (13), after some transformation, we arrive at a functional relationship between the functions $\nu(x)$ and $\tau(x)$, obtained from the domain D_2

$$\nu(x) = \int\limits_{0}^{x} K_2(x,\xi)\,\tau(\xi)d\xi + f_1(x), \tag{14}$$

where $K_2(x,\xi) = [a_\xi(\xi,0) - c(\xi,0)]\Gamma(x,\xi) + a(\xi,0)\Gamma_\xi(x,\xi)$, $f_1(x) = a(0,0)\psi_1(0)\Gamma(x,0) + \Psi(x)$. Eliminating $\nu(x)$ from (8) and (14), we obtain an integro-differential equation for determining $\tau(x)$

$$\tau'''(x) + a(x,0)\tau(x) = \int\limits_{0}^{x} K_{2x}(x,\xi)\,\tau(\xi)d\xi + f_1'(x). \tag{15}$$

Let's represent the equation (15) in the form

$$\tau'''(x) = F(x,\tau), \tag{16}$$

where

$$F(x,\tau) \equiv -a(x,0)\tau(x) + \int\limits_{0}^{x} K_{2x}(x,\xi)\,\tau(\xi)d\xi + f_1'(x).$$

We introduce a new function $\tau_1(x)$:

$$\tau(x) = \tau_1(x) + z(x), \tag{17}$$

where $z(x) = [\varphi_2(0) - \varphi_1(0) - \varphi_3(0)\ell]\frac{x^2}{\ell^2} + \varphi_3(0)x + \varphi_1(0)$. Then, for $\tau_1(x)$ we obtain the following boundary value problem

$$\tau_1'''(x) = F(x,\tau), \tag{18}$$

$$\tau_1(0) = 0, \quad \tau_1(\ell) = 0, \quad \tau_1'(0) = 0. \tag{19}$$

It is not difficult to prove that the homogeneous problem (18) and (19) has only a trivial solution. Therefore, there is a unique Green's function for this problem. To obtain an explicit solution to the problem (18) and (19) we will use the Green's function method [24]. We will look for the Green's function in the form

$$G_1(x,\xi) = \begin{cases} \alpha_1 x^2 + \beta_1 x + \gamma_1, & 0 \le x \le \xi, \\ \alpha_2 x^2 + \beta_2 x + \gamma_2, & \xi \le x \le \ell, \end{cases} \tag{20}$$

where α_i, β_i, $\gamma_i\,(i=1,2)$ are yet unknown functions of ξ. The function $G_1(x,\xi)$ must satisfy the following boundary conditions

$$G_1(0,\xi) = 0, \quad G_1(\ell,\xi) = 0, \quad G_{1x}(0,\xi) = 0 \tag{21}$$

and pairing (gluing) conditions

$$G_1(\xi+0,\xi) - G_1(\xi-0,\xi) = 0, \quad G_{1x}(\xi+0,\xi) - G_{1x}(\xi-0,\xi) = 0, \tag{22}$$

$$G_{1xx}(\xi+0,\xi) - G_{1xx}(\xi-0,\xi) = 1. \tag{23}$$

Using the conditions (21), (22), and (23), we determine the coefficients

$$\alpha_1 = -\frac{(\xi-\ell)^2}{2\ell^2}, \quad \beta_1 = 0, \quad \gamma_1 = 0, \quad \alpha_2 = \frac{2\ell\xi - \xi^2}{2\ell^2}, \quad \beta_2 = -\xi, \quad \gamma_2 = \frac{\xi^2}{2}.$$

Therefore, taking into account (20), the Green's function has the form

$$G_1(x,\xi)=\begin{cases}-\frac{x^2}{2\ell^2}(\ell-\xi)^2, & 0\le x\le\xi,\\ \frac{\xi(\ell-x)}{2\ell^2}(x\xi+\ell\xi-2\ell x), & \xi\le x\le\ell.\end{cases}$$

The Green's function can be represented as [22]

$$G_1(x,\xi)=\begin{cases}\frac{\xi(\ell-x)}{2\ell^2}(x\xi+\ell\xi-2\ell x), & 0\le\xi\le x,\\ -\frac{x^2}{2\ell^2}(\ell-\xi)^2, & x\le\xi\le\ell.\end{cases}$$

Thus, the solution to the problem (18) and (19) has the form

$$\tau_1(x)=\int\limits_0^\ell G_1(x,\xi)F(\xi,\tau)d\xi. \tag{24}$$

Then, from (17) taking into account (24), for $\tau(x)$ we arrive at the Fredholm integral equation of the second kind

$$\tau(x)=\int\limits_0^\ell K(x,\xi)\tau(\xi)d\xi+z_1(x), \tag{25}$$

where

$$K(x,\xi)=-a(\xi,0)G_1(x,\xi)+\int\limits_\xi^\ell G_1(x,t)K_{2x}(t,\xi)dt,\quad z_1(x)=z(x)+\int\limits_0^\ell G_1(x,\xi)f_1'(\xi)d\xi.$$

Let $||K||=\max_{0\le x,\xi\le\ell}|K(x,\xi)|$. If

$$||K||<1, \tag{26}$$

then the solution to the equation (25) exists and is unique [24]. Thus, $\tau(x)$ is completely defined.

3. CONSTRUCTING A SOLUTION TO A PROBLEM IN THE DOMAIN D_1

After defining $\tau(x)$ to construct a solution to Problem 1 in the domain D_1, consider the following auxiliary problem.

Problem 1.1. Find in the domain D_1 the function $u(x,y)\in C(\bar{D}_1)\cap C^1(D_1)\cap C^{3,1}(D_1)$, satisfying the equation (1), the boundary conditions (4), (5) and the initial condition

$$u(x,0)=\tau(x),\quad 0\le x\le\ell. \tag{27}$$

We introduce the denotation

$$u_x(x,y)=\upsilon(x,y),\quad (x,y)\in D_1. \tag{28}$$

Let

$$\upsilon(\ell,y)=\varphi(y),\quad 0\le y\le h, \tag{29}$$

where $\varphi(y)$ is yet unknown function. Then, taking into account (28), from the equation (1) we have

$$\upsilon_{xx}-\upsilon_y=0,\quad (x,y)\in D_1. \tag{30}$$

The solution to the equation (30), satisfying the conditions (5), (29) and $\upsilon(x,0)=\tau'(x)$ has the form

$$\upsilon(x,y)=\int\limits_0^y G_\xi(x,y;0,\eta)\varphi_3(\eta)d\eta-\int\limits_0^y G_\xi(x,y;\ell,\eta)\varphi(\eta)d\eta$$

$$+\int_0^{\ell} G(x,y;\xi,0)\,\tau'(\xi)\,d\xi, \quad (x,y)\in D_1, \tag{31}$$

where

$$G(x,y;\xi,\eta)=\frac{1}{2\sqrt{\pi(y-\eta)}}\sum_{n=-\infty}^{n=+\infty}\left[\exp\left(-\frac{(x-\xi+2n\ell)^2}{4(y-\eta)}\right)-\exp\left(-\frac{(x+\xi+2n\ell)^2}{4(y-\eta)}\right)\right]$$

is Green function [23].

Integrating (31) over x in the range from 0 to x and taking into account the first condition (4), we have

$$u(x,y)=\varphi_1(y)+\int_0^x dt\int_0^y G_\xi(t,y;0,\eta)\,\varphi_3(\eta)d\eta-\int_0^x dt\int_0^y G_\xi(t,y;\ell,\eta)\,\varphi(\eta)d\eta$$
$$+\int_0^x dt\int_0^{\ell} G(t,y;\xi,0)\,\tau'(\xi)\,d\xi, \quad (x,y)\in D_1. \tag{32}$$

From here, using the second condition (4), we get

$$\int_0^y\left[\int_0^{\ell} G_\xi(t,y;\ell,\eta)\,dt\right]\varphi(\eta)d\eta=\Phi(y), \tag{33}$$

where

$$\Phi(y)=\varphi_1(y)-\varphi_2(y)+\int_0^{\ell} dt\int_0^y G_\xi(t,y;0,\eta)\,\varphi_3(\eta)d\eta+\int_0^{\ell} dt\int_0^{\ell} G(t,y;\xi,0)\,\tau'(\xi)\,d\xi.$$

It's easy to see that

$$G_\xi(t,y;\ell,\eta)=-\frac{1}{\sqrt{\pi(y-\eta)}}\frac{\partial}{\partial t}\left\{\exp\left[-\frac{(t-\ell)^2}{4(y-\eta)}\right]\right\}+\frac{1}{4\sqrt{\pi(y-\eta)^3}}$$
$$\times\left\{\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty}[t+(2n-1)\ell]\exp\left[-\frac{[t+(2n-1)\ell]^2}{4(y-\eta)}\right]\right.$$
$$\left.+\sum_{\substack{n=-\infty\\ n\neq -1}}^{+\infty}[t+(2n+1)\ell]\exp\left[-\frac{[t+(2n+1)\ell]^2}{4(y-\eta)}\right]\right\}.$$

Then, we have

$$\int_0^{\ell} G_\xi(t,y;\ell,\eta)dt=-\frac{1}{\sqrt{\pi}\sqrt{y-\eta}}+\frac{1}{\sqrt{\pi}}N_1(y,\eta), \tag{34}$$

where

$$N_1(y,\eta)=\frac{1}{\sqrt{y-\eta}}\exp\left(-\frac{\ell^2}{4(y-\eta)}\right)+\frac{1}{4\sqrt{(y-\eta)^3}}$$
$$\times\int_0^{\ell}\left\{\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty}[t+(2n-1)\ell]\exp\left[-\frac{[t+(2n-1)\ell]^2}{4(y-\eta)}\right]\right.$$

$$+\sum_{\substack{n=-\infty\\ n\neq -1}}^{+\infty}[t+(2n+1)\ell]\exp\left[-\frac{[t+(2n+1)\ell]^2}{4(y-\eta)}\right]\Bigg\}dt.$$

Taking into account (34), we write the equation (33) in the form

$$\int\limits_0^y \frac{\varphi(\eta)}{\sqrt{y-\eta}}d\eta=\int\limits_0^y N_1(y,\eta)\varphi(\eta)d\eta-\sqrt{\pi}\Phi(y), \tag{35}$$

Inverting the equation (35) as the Abel integral equation, we obtain the Volterra integral equation of the 2nd kind [24]

$$\varphi(y)=\int\limits_0^y N(y,\eta)\varphi(\eta)d\eta+\Phi_1(y),$$

which admits a unique solution, where

$$N(y,\eta)=\frac{1}{\pi}\int\limits_\eta^y \frac{N_{1t}(t,\eta)}{\sqrt{y-t}}dt,\quad \Phi_1(y)=-\frac{1}{\sqrt{\pi}}\int\limits_0^y \frac{\Phi'(\eta)}{\sqrt{y-\eta}}d\eta.$$

After determining $\varphi(y)$, the solution to Problem 1 is determined by the formula (32).

4. CONSTRUCTING A SOLUTION TO A PROBLEM IN THE DOMAIN D_2

After defining $\tau(x)$, solving Problem 1 in the domain D_2 is reduced to solving the following auxiliary problem.

Problem 1.2. Find in the domain D_2 the function $u(x,y)\in C(\bar{D}_2)\cap C^1(D_2)\cap C^{2,1}(D_2)$, satisfying the equation (2), the boundary conditions (6) and the initial condition (27).

To obtain solutions to Problem 1.2, we integrate the equation (2) over x twice in the range from 0 to x and taking into account the boundary conditions (6), we obtain

$$u_y(x,y)=g_1(x,y)+\int\limits_0^x P_1(x,y,\xi)u_y(\xi,y)d\xi+\int\limits_0^x Q_1(x,y,\xi)u(\xi,y)d\xi, \tag{36}$$

where

$$g_1(x,y)=\chi_1'(y)+\chi_2'(y)x+a(0,y)\chi_1(y)x,\quad P_1(x,y,\xi)=(\xi-x)b(\xi,y),$$

$$Q_1(x,y,\xi)=(x-\xi)[a_\xi(\xi,y)-c(\xi,y)]-a(\xi,y).$$

Let $R_1(x,y,\xi)$ be resolvent of the kernel $P_1(x,y,\xi)$. Inverting the integral equation (36) with respect to $u_y(x,y)$, we have

$$u_y(x,y)=g(x,y)+\int\limits_0^x Q(x,y,\xi)u(\xi,y)d\xi, \tag{37}$$

where

$$g(x,y)=g_1(x,y)+\int\limits_0^x R_1(x,y,t)g_1(t,y)dt,$$

$$Q(x,y,\xi)=Q_1(x,y,\xi)+\int\limits_\xi^x R_1(x,y,t)Q_1(t,y,\xi)dt.$$

Integrating (37) over y, ranging from y to 0, we obtain the Volterra integral equation of the 2-nd kind with two independent variables

$$u(x,y) = u_0(x,y) + \int\limits_0^x d\xi \int\limits_0^y Q(x,\eta,\xi)u(\xi,\eta)d\eta, \tag{38}$$

where

$$u_0(x,y) = \tau(x) + \int\limits_0^y g(x,\eta)d\eta.$$

Using the method of successive approximations [24], we will find a unique solution to the equation (38) and thereby find a solution to Problem 1.2. Thus, it is proven

Theorem 1. *If the conditions (3), (7), and (26) are met, then the solution to problem 1 exists and is unique.*

5. PROBLEM STATEMENT 2

Let D denote the same domain described in Section 1.

Problem 2. Find in the domain D a function $u(x,y)$ from the class $C(\bar{D}) \cap C^1(D) \cap \cap [C^{3,1}(D_1) \cup C^{2,1}(D_2)]$, satisfying the equation (1) in the domain D_1 and the boundary conditions

$$u(0,y) = \varphi_1(y), \quad 0 \le y \le h,$$
$$u_x(0,y) = \varphi_3(y), \quad 0 \le y \le h, \quad u_x(\ell,y) = \varphi_4(y), \tag{39}$$

satisfying the equation (2) in the domain D_2 and the boundary conditions

$$u(0,y) = \psi_1(y), \quad u_x(0,y) = \psi_2(y), \quad -h_1 \le y \le 0,$$

where $\varphi_i(y)$ $(i = \overline{1,3})$ and $\psi_j(y)$ $(j = 1,2)$ are given functions and

$$\varphi_1(0) = \psi_1(0), \quad \varphi_3(0) = \psi_2(0). \tag{40}$$

To solve Problem 2, we will use the same notation used to solve Problem 1. Passing to the limit at $y \to +0$ from the equation (1) we obtain the relation (8), and for $\tau(x)$—matching conditions

$$\tau(0) = \varphi_1(0), \quad \tau'(0) = \varphi_3(0), \quad \tau'(\ell) = \varphi_4(0).$$

The relation obtained from the region D_2, as in Problem 1, has the form (9), and the matching conditions will take the form (11). Excluding $\nu(x)$ from (8) and (14), to determine $\tau(x)$ we obtain the equation (16).

By introducing a new function $\tau_2(x)$ according to the formula $\tau(x) = \tau_2(x) + \omega(x)$, where $\omega(x) = [\varphi_4(0) - \varphi_3(0)]\frac{x^2}{2\ell} + \varphi_3(0)x + \varphi_1(0)$, we arrive at the following problem

$$\tau_2'''(x) = F(x,\tau), \tag{41}$$

$$\tau_2(0) = 0, \quad \tau_2'(0) = 0, \quad \tau_2'(\ell) = 0. \tag{42}$$

To obtain a representation of the solution to the problem (41) and (42), we construct the Green's function $G_2(x,\xi)$, which must satisfy the following conditions

$$G_2(0,\xi) = 0, \quad G_{2x}(0,\xi) = 0, \quad G_{2x}(\ell,\xi) = 0,$$

$$G_2(\xi+0,\xi) - G_2(\xi-0,\xi) = 0, \quad G_{2x}(\xi+0,\xi) - G_{2x}(\xi-0,\xi) = 0,$$

$$G_{2xx}(\xi+0,\xi) - G_{2xx}(\xi-0,\xi) = 1.$$

It is easy to check that such a function exists and it has the form

$$G_2(x,\xi) = \begin{cases} \frac{\xi}{2\ell}(x^2 - 2\ell x + \ell\xi), & 0 \le \xi \le x, \\ -\frac{x^2(\ell-\xi)}{2\ell}, & x \le \xi \le \ell. \end{cases}$$

Thus, the solution to the problem (41) and (42) can be represented in the form

$$\tau_2(x) = \int\limits_0^\ell G_2(x,\xi)F(\xi,\tau)d\xi.$$

Then, for $\tau(x)$ we arrive at the Fredholm integral equation of the second kind

$$\tau(x) = \int\limits_0^\ell H(x,\xi)\tau(\xi)d\xi + \omega_1(x), \tag{43}$$

where

$$H(x,\xi) = -a(\xi,0)G_2(x,\xi) + \int\limits_\xi^\ell G_2(x,t)K_{2t}(t,\xi)dt, \quad \omega_1(x) = \omega(x) + \int\limits_0^\ell G_2(x,\xi)f_1'(\xi)d\xi.$$

Let $||H|| = \max_{0\le x,\xi\le\ell} |H(x,\xi)|$. If

$$||H|| < 1, \tag{44}$$

then the solution to the equation (43) exists and is unique [25].

After defining $\tau(x)$, to solve Problem 2 in the region D_1, we introduce the notation

$$u_x(x,y) = w(x,y), \quad (x,y) \in D_1. \tag{45}$$

Then, for $w(x,y)$ we arrive at the first boundary value problem

$$w_{xx} - w_y = 0, \quad (x,y) \in D_1,$$
$$w(0,y) = \varphi_3(y), \quad w(\ell,y) = \varphi_4(y), \quad 0 \le y \le h,$$
$$w(x,0) = \tau'(x), \quad 0 \le x \le \ell,$$

which solution has the form

$$w\,(x,y) = \int\limits_0^y G_\xi\,(x,y;0,\eta)\,\varphi_3(\eta)d\eta - \int\limits_0^y G_\xi\,(x,y;\ell,\eta)\,\varphi_4(\eta)d\eta + \int\limits_0^\ell G\,(x,y;\xi,0)\,\tau'\,(\xi)\,d\xi, \tag{46}$$

where $G(x,y;\xi,\eta)$ is the Green's function [26]. Integrating the relation (46) over x in the range from 0 to x and taking into account the notation (45), as well as the condition (39), we find the solution to Problem 2 in the region D_1 as

$$u\,(x,y) = \varphi_1(y) + \int\limits_0^x w(\xi,y)d\xi, \quad (x,y) \in D_1.$$

The solution to Problem 2 in the domain D_2 is defined in the same way as the solution to Problem 1.2. Thus, it is proven

Theorem 2. *If the conditions (3), (40), and (44) are met, then the solution to problem 2 exists and is unique.*

REFERENCES

1. S. D. Algazin and I. A. Kiyko, *Flutter of Plates and Shells* (Nauka, Moscow, 2006) [in Russian].
2. M. Kh. Shkhanukov, "On some boundary value problems for a third-order equation arising when modelling fluid filtration in porous media," Differ. Uravn. **18**, 689–699 (1982).
3. T. D. Dzhuraev and J. Popelek, "On the classification and reduction to canonical form of third-order partial differential equations," Differ. Equat. **27**, 1225–1235 (1991).
4. F. Tricomi, "Sulle equazioni lineari alle derivate parziali del secondo ordine di tipo misto," Atti R. Accad. Naz. Lincei, Mem. Cl. Sci. Fis. Mat. Nat. (V) **14**, 133–247 (1923).

5. T. D. Dzhuraev, *Boundary Value Problems for Equations of Mixed and Mixed-Composite Types* (Fan, Tashkent, 1979) [in Russian].
6. T. D. Dzhuraev, A. Sopuev, and M. Mamazhanov, *Boundary Value Problems for Equations of Parabolic-Hyperbolic Type* (Fan, Tashkent, 1986) [in Russian].
7. A. M. Nakhushev, *Equations of Mathematical Biology* (Vyssh. Shkola, Moscow, 1995) [in Russian].
8. V. I. Zhegalov and E. A. Utkina, "On a third-order pseudoparabolic equation," Russ. Math. (Izv. VUZ) **43** (10), 70–73 (1999).
9. T. D. Dzhuraev and Yu. P. Apakov, "On a self-similar solution of one third-order equation with multiple characteristics," Vestn. Samar. Tekh. Univ., Ser.: Fiz.-Mat. Nauki **2** (15), 18–26 (2007).
10. T. D. Dzhuraev and Yu. P. Apakov, "On the theory of a third-order equation with multiple characteristics containing the second time derivative," Ukr. Math. J. **62**, 40–51 (2010).
11. Yu. P. Apakov, "On the solution of a boundary-value problem for a third-order equation with multiple characteristics," Ukr. Math. J. **64**, 1–11 (2012).
12. K. B. Sabitov, *Direct and Inverse Problems for Equations of Mixed Parabolic-Hyperbolic Type* (Nauka, Moscow, 2016) [in Russian].
13. T. K. Yuldashev, "Inverse problem for a nonlinear Benney–Luke type integro-differential equations with degenerate kernel," Russ. Math. (Izv. VUZ) **60** (8), 53–60 (2016).
14. T. K. Yuldashev, "Solvability of a boundary value problem for a differential equation of the Boussinesq type," Differ. Equat. **54**, 1384–1393 (2018).
15. T. K. Yuldashev and B. J. Kadirkulov, "Nonlocal problem for a mixed type fourth-order differential equation with Hilfer fractional operator," Ural Math. J. **6**, 153–167 (2020).
16. T. K. Yuldashev and B. J. Kadirkulov, "Inverse boundary value problem for a fractional differential equations of mixed type with integral redefinition conditions," Lobachevskii J. Math. **42**, 649–662 (2021).
17. T. K. Yuldashev, B. I. Islomov, and E. K. Alikulov, "Boundary-value problems for loaded third-order parabolic-hyperbolic equations in infinite three-dimensional domains," Lobachevskii J. Math. **41**, 926–944 (2020).
18. Yu. P. Apakov, *On the Theory of Third-Order Equations with Multiple Characteristics* (Fan va texnologiya, Tashkent, 2019) [in Russian].
19. Yu. P. Apakov, "On unique solvability of boundary-value problem for a viscous transonic equation," Lobachevskii J. Math. **41**, 1754–1761 (2020).
20. A. Sopuev, Yu. P. Apakov, and O. M. Mirzaev, "Solution of the second boundary value problem for a fifth-order equation with multiple characteristics," Vestn. OshGU, №1, 136–148 (2022).
21. Yu. P. Apakov and R. A. Umarov, "Solution of the boundary value problem for a third order equation with little terms. Construction of the Green's function," Lobachevski J. Math. **43**, 738–748 (2022). http://doi.org/10.1134/S199508022206004X
22. Yu. P. Apakov and A. A. Hamitov, "Third boundary value problem for an equation with the third order multiple characteristics in three dimensional space," Lobachevskii J. Math. **44**, 523–532 (2023). http://doi.org/10.1134/S1995080223020099
23. T. K. Yuldashev, Yu. P. Apakov, and A. Kh. Zhuraev, "Boundary value problem for third order partial integro-differential equation with a degenerate kernel," Lobachevskii J. Math. **42**, 1317–1327 (2021). http://doi.org/10.1134/S1995080221060329
24. M. L. Krasnov, A. I. Kiselev, and G. I. Makarenko, *Integral Equations: Problems and Examples with Detailed Solutions* (Editorial URSS, Moscow, 2003) [in Russian].
25. K. G. Kozhobekov, "Nonlocal conjugation problem for nonlinear partial differential equations of third order," Vestn. KazNU, Ser. Mat., Mekh., Inform. **60** (1), 3–40 (2009).
26. A. D. Polyanin, *Handbook of Linear Equations of Mathematical Physics* (Fizmatlit, Moscow, 2001) [in Russian].